

Noncommutative spaces and matrix embeddings on flat \mathbb{R}^{2n+1}

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ABSTRACT: We conjecture an embedding operator which assigns, to any $2n + 1$ hermitian matrices, a $2n$ -dimensional hypersurface in flat $(2n + 1)$ -dimensional Euclidean space. This corresponds to precisely defining a fuzzy $D(2n)$ -brane corresponding to N $D0$ -branes. Points on the emergent hypersurface correspond to zero eigenstates of the embedding operator, which have an interpretation as coherent states underlying the emergent noncommutative geometry. Using this correspondence, all physical properties of the emergent $D(2n)$ -brane can be computed. We apply our conjecture to noncommutative flat and spherical spaces. As a by-product, we obtain a construction of a rotationally symmetric flat noncommutative space in 4 dimensions.

KEYWORDS: D-branes, Non-Commutative Geometry

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1 Introduction and conjecture

The appearance of matrix coordinates, where the positions of N identical objects in d dimensions are described by d $N \times N$ matrices instead of N d -vectors, is common in string theory. Geometric interpretation of non-commuting matrix coordinates often involves an emergent higher dimensional object. The exact shape and other properties of this emergent object can be hard to study; outside of highly symmetric surfaces such as spheres, only some approximate methods (such as diagonalizing the matrices one at a times) are usually employed. In [1], a method for determining a surface embedded in \mathbb{R}^3 and associated with any three matrices was given, providing a concrete solution to this problem. In [2], the geometry of this surface was examined in detail, proving the correspondence principle between matrix commutators and a Poisson structure on the emergent surface.

It is natural to ask about generalizing these results to higher dimensions. Higher dimensional noncommutative spaces posses a much richer phenomenology than noncommutative surfaces do and an explicit embedding into flat space would make their study easier and more concrete. Below, in equation (1.2), we conjecture an embedding operator which makes this possible for even-dimensional noncommutative hypersurfaces embedded in a odd-dimensional flat space.

In their paper, [1] use a probe brane interacting with a stack of N D0-branes at an orbifold point in the BFSS model, reducing the dimension of the space transverse to the D0-branes to three. The emergent surface is defined as the locus of possible positions for the probe brane where a fermionic string stretched from the stack to the probe brane has a massless mode. The fermion mass matrix is given by the following effective Hamiltonian:

$$H_{\text{eff}}(x_i) = \sum_{i=1,2,3} \sigma^i \otimes (X_i - x_i) , \quad (1.1)$$

where σ^i are Pauli matrices, X_i are Hermitian $N \times N$ matrices corresponding to the positions of the stack of D0-branes and x_i are the positions of the probe brane. The stretched string has a zero mass fermionic mode when H_{eff} has a zero eigenvalue. Thus, the surface corresponding to the three matrices X_i is given by the locus of points where H_{eff} has zero eigenvalues, defining a co-dimension one surface in flat \mathbb{R}^3 .

H_{eff} above plays a role of an ‘embedding operator’: it specifies how the emergent surface given by three matrices X_i should be embedded in flat \mathbb{R}^3 . Since equation (1.1) was obtained from an orbifold construction, with Pauli matrices arising from a dimensional reduction of Dirac Γ matrices from 9 dimensions to 3, a natural guess for the generalization of the embedding operator to arbitrary odd dimensions is

$$E_d(x_i) = \sum_{i=1}^d \gamma^i \otimes (X_i - x_i) , \quad (1.2)$$

where γ^i are the (Euclidean) Dirac matrices in d dimensions, which form a representation of the Clifford algebra

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij} . \quad (1.3)$$

We have introduced a new symbol, E_d , to denote the embedding operator in \mathbb{R}^d . For $d = 9$ this operator has been used in [3] to study thermal configurations in the BFSS model. Similar Dirac operators have been used in [4] to define the location of D-brane intersections and the resulting emergent gravity.

As we will see, our conjectured embedding operator ‘knows’ a lot about noncommutative geometry. For example, a noncommutative sphere S^{2d} with $\text{SO}(2d+1)$ symmetry cannot locally (near some point p) look like the standard flat noncommutative space, since the latter is never fully rotationally symmetric, while S^{2d} should retain $\text{SO}(2d)$ symmetry around point p . Examining the kernel of the embedding operator E_d for a noncommutative four-sphere we find an auxiliary spin space whose presence restores $\text{SO}(4)$ invariance, resolving the puzzle.

It would be very interesting to obtain formula (1.2) from string theory considerations. For $d = 5$ and $d = 7$, the computation might proceed along the lines of [1], using an orbifold. For $d = 9$, another method might be more applicable (see the discussion in [1]).

The remainder of this paper is organized as follows: in the next section, we set conventions and observe that once E_d is known in some odd dimension d , it is possible to obtain the embedding operators in all lower dimensions by simply setting some of the matrices to zero, two at a time. In section 3, we discuss flat noncommutative space and generalize most of our results from [2] to higher dimensions. In section 4 we study the noncommutative four-sphere embedded in \mathbb{R}^5 , in particular obtaining a flat noncommutative space with $\text{SO}(4)$ rotational symmetry as an approximation to the sphere on a small patch. In section 5, we discuss further examples of four dimensional noncommutative surfaces. Finally, in section 6 we try to study even dimensions by setting just one of the matrices to zero. That this naive guess fails to work can be demonstrated by considering the noncommutative three-sphere, S^3 .

2 Conventions and a recursive property of E_d

Our embedding operators have the property that once E_d is known in some odd dimension d , it is possible to obtain the embedding operators in all lower dimensions recursively. The easiest way to see that our family of embedding operators has this property is to use an iterative definition of the γ matrices as follows.¹ In $d = 1$, we trivially take γ^1 to be the 1×1 unit matrix. Then, denoting the γ matrices in $d - 2$ dimensions with $\tilde{\gamma}^i$, we have in d dimensions that

$$\gamma^i = \sigma^3 \otimes \tilde{\gamma}^i, \quad i = 1, \dots, d-2, \quad (2.1)$$

$$\gamma^{d-1} = \sigma^1 \otimes \mathbf{1}, \quad (2.2)$$

$$\gamma^d = \sigma^2 \otimes \mathbf{1}. \quad (2.3)$$

The dimension of the γ matrices is thus $2^n = 2^{(d-1)/2}$. For $d = 3$, we obtain a permutation of the Pauli matrices: $\gamma^1 = \sigma^3$, $\gamma^2 = \sigma^1$, $\gamma^3 = \sigma^2$.

Now, in some odd number of dimensions d set the last two matrices X_{d-1} and X_d to zero. We can then reduce E_d to E_{d-2} : if $X_{d-1} = X_d = \mathbf{0}$, then

$$E_d(x_1, \dots, x_d) = \sum_{i=1}^{d-2} (\sigma^3 \otimes \tilde{\gamma}^i) \otimes (X_i - x_i) - (\sigma_1 \otimes \mathbf{1}) \otimes (x_{d-1}) - (\sigma_2 \otimes \mathbf{1}) \otimes (x_d) \quad (2.4)$$

One can show that for the above operator to have a zero eigenvector, we must necessarily have $x_{d-1} = x_d = 0$. Then, the operator above can be reduced to

$$\sigma^3 \otimes E_{d-2}(x_1, \dots, x_{d-2}). \quad (2.5)$$

Thus, once a construction of E_d is known in some odd number of dimensions, it is easy to construct all the smaller odd dimensional cases. In section 6 we will discuss our attempt to obtain an embedding operator in an even number of dimensions by setting just one of the matrices to zero.

To write the γ matrices in an explicit form it is convenient to introduce the following notation:

$$\sigma_n(c_1, \dots, c_n) := \sigma^{c_1} \otimes \dots \otimes \sigma^{c_n}, \quad (2.6)$$

where the coefficients c_i take integer values from 0 to 3 and where we define $\sigma^0 = \mathbf{1}$. In this notation, the recursive definition of γ matrices implies that

$$\begin{aligned} \gamma^1 &= \sigma_n(3, 3, 3, \dots, 3, 3, 3) \\ \gamma^2 &= \sigma_n(3, 3, 3, \dots, 3, 3, 1) \\ \gamma^3 &= \sigma_n(3, 3, 3, \dots, 3, 3, 2) \\ \gamma^4 &= \sigma_n(3, 3, 3, \dots, 3, 1, 0) \\ \gamma^5 &= \sigma_n(3, 3, 3, \dots, 3, 2, 0) \\ &\vdots \\ \gamma^{d-1} &= \sigma_n(1, 0, 0, \dots, 0, 0, 0) \\ \gamma^d &= \sigma_n(2, 0, 0, \dots, 0, 0, 0) \end{aligned}$$

¹We follow here [5].

To complete our conventions, we make the following choice for the Pauli matrices:

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \sigma^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \quad (2.7)$$

3 Noncommutative \mathbb{R}^{2n}

As the first example, set $X^1 = \mathbf{0}$ and consider the other $d - 1$ matrices to have a commutation relation

$$[X_i, X_j] = i\theta_{ij} \text{ for } i, j = 2, \dots, d. \quad (3.1)$$

This, of course, is simply flat noncommutative space, extending in dimensions 2 through d (assuming θ has full rank). θ is an antisymmetric even dimensional matrix which can be, by an orthogonal change of basis and therefore without loss of generality, brought into the block-diagonal form

$$\theta = \text{diag} \left(\begin{bmatrix} 0 & \theta_1 \\ -\theta_1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & \theta_n \\ -\theta_n & 0 \end{bmatrix} \right). \quad (3.2)$$

We define $A_a = X_{2a} + iX_{2a+1}$ for $a = 1, \dots, n$. A_a and A_a^\dagger are the lowering and raising operators of a harmonic oscillator with $[A_a, A_a^\dagger] = 2\theta_a$. The lowering operators A_a have eigenstates $|\alpha\rangle_a$ (the coherent states), corresponding to every complex number α : $A_a|\alpha\rangle_a = \alpha|\alpha\rangle_a$. E_d can be written as

$$\sum_{a=1}^n \left(\Lambda_-^a \otimes (A_a - \alpha_a) + \Lambda_+^a \otimes (A_a^\dagger - \bar{\alpha}_a) \right), \quad (3.3)$$

where

$$\Lambda_\pm^a = \sigma_n(\underbrace{3, \dots, 3}_{n-a \text{ times}}, \pm, \underbrace{0, \dots, 0}_{a-1 \text{ times}}) = \gamma_{2a} \pm i\gamma_{2a+i}. \quad (3.4)$$

In this form, it is easy to see that

$$|\Lambda(\alpha)\rangle = \left(\bigotimes_{a=1}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \otimes \left(\bigotimes_{a=1}^n |\alpha\rangle_a \right) \quad (3.5)$$

is a zero eigenvector for E_d at a point given by $x_1 = 0$ and $x_{2a} + ix_{2a+1} = \alpha_a$. Thus there is a zero eigenvector for every point on the co-dimension one hypersurface given by $x_1 = 0$. The first factor in the above zero eigenvector is simply one of the highest weight vectors of the Clifford algebra selected by the particular form of raising operators Λ_+ which we are using. We will denote it with V_d :

$$V_d := \left(\bigotimes_{a=1}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right). \quad (3.6)$$

V_d is an eigenvector of γ^1 with eigenvalue 1 and has the property that $\Lambda_+^a V_d = 0$ for all a .

We can expect many noncommutative spaces to have the property that the embedding operator has a single zero eigenvector at a given point on the emergent surface. Those spaces should, locally, look like noncommutative flat space given by equation (3.1). We will see in the next section that, for $d > 3$, non-degenerate noncommutative spaces exist whose embedding operators have multiple zero eigenvectors at a point. However, for those that don't, our work [2] on emergent surfaces in the large N limit can easily be generalized to higher dimensions. Once we consider compact surfaces with curvature, the commutation relation in equation (3.1) are only true approximately and can be satisfied by finite matrices (for details, see [2]). Similar results have been obtained before in [6] (see also [7] and the references therein).

In the rest of this section, we state the salient results and conjectures.

Assume, then, that the zero eigenvector $|\Lambda_p\rangle$ of the embedding operator is unique at every point p of the emergent surface. The normal vector to this surface at point p is given by²

$$n_i = \langle \Lambda_p | \gamma_i \otimes \mathbf{1} | \Lambda_p \rangle. \quad (3.7)$$

For simplicity, we now rotate our surface so that the normal vector at the point of interest points in the x_1 direction. We conjecture that the eigenvector is equal to, approximately, a product of an appropriate highest weight state V_d^θ and a N -dimensional vector:

$$|\Lambda\rangle = V_d^\theta \otimes |\alpha\rangle + \text{corrections that vanish for } N \rightarrow 0. \quad (3.8)$$

Further, we can define a local noncommutativity matrix θ_{ij} at point p by

$$\theta_{ij} = \langle \alpha | -i[X_i, X_j] | \alpha \rangle, \quad \text{for } i, j = 2, \dots, d. \quad (3.9)$$

θ_{ij} is an antisymmetric two-form on the emergent surface; it defines a Poisson bracket of two functions f and h :

$$\{f, h\} := \frac{N\theta_{ab}}{\sqrt{\det g}} \partial_a f \partial_b h, \quad (3.10)$$

where g_{ab} is the pullback of the flat metric on \mathbb{R}^d to the $d-1$ dimensional emergent space. From this Poisson bracket, we divide the $d-1$ directions x_2, \dots, x_d into raising and lowering operators just like we did above. In particular, we have a new set of lowering and raising operators on the spinor space, $\Lambda_\pm^{\theta,a}$ (defined, in a particular basis, in equation (3.4)). The highest weight state in equation (3.8) has $\Lambda_+^{\theta,a} V_d^\theta = 0$.

The N -dimensional state $|\alpha\rangle$ should be interpreted as a coherent state associated with the point p . Since $(E_d)^2 |\Lambda\rangle = 0$, we have

$$\langle \Lambda_p | \mathbf{1} \otimes \sum_i (X_i - x_i)^2 | \Lambda_p \rangle = -\frac{1}{2} \langle \Lambda_p | \sum_{i \neq j} (\gamma_i \gamma_j) \otimes [X_j, X_k] | \Lambda_p \rangle. \quad (3.11)$$

Substituting the factorization condition (3.8), we obtain

$$\langle \alpha | \sum_i (X_i - x_i)^2 | \alpha \rangle = -\frac{1}{2} \langle \alpha | \sum_{i \neq j} (n_{ij} [X_i, X_j]) | \alpha \rangle, \quad (3.12)$$

²The arguments for this and other statements below are basically identical to that given in [2] for $d = 3$.

where the two-form n_{ij} is defined below, in equation (3.15). However, since our noncommutative space is a direct product of n copies of two dimensional noncommutative space, a better way to study the properties of the coherent state is work in a basis where the noncommutativity is given by equation (3.2) and to write $|\alpha\rangle$ as a product of n coherent states $|\alpha\rangle = |\alpha\rangle_1 \otimes \dots \otimes |\alpha\rangle_n$.

Once we have coherent states $|\alpha_p\rangle$ corresponding to every point p on the surface, we can associate any $N \times N$ matrix M with functions on the surface, via $M \rightarrow \langle \alpha_p | M | \alpha_p \rangle$. This is the Berezin approach to noncommutative geometry [8]. It gives a natural map between commutators of operators and an antisymmetric Lie bracket on the surface. This bracket turns out to be equal to the Poisson bracket defined in equation (3.10) as long as, in addition to the factorization condition (3.8), we also have that

$$\langle \alpha | -i[X_j, X_1] | \alpha \rangle, \quad \text{for } i, j = 2, \dots, d, \quad (3.13)$$

is much smaller than $\|\theta_{ij}\|$ for $N \rightarrow \infty$.

It is useful to define two antisymmetric two-forms on \mathbb{R}^d :

$$\hat{\theta}_{ij} = \langle \alpha | -i[X_i, X_j] | \alpha \rangle, \quad \text{for } i, j = 1, \dots, d, \quad (3.14)$$

and

$$n_{ij} = \frac{1}{2} \langle V_d^\theta | i[\gamma_i, \gamma_j] | V_d^\theta \rangle \quad \text{for } i \neq j. \quad (3.15)$$

It is easy to see that $n_{1k} = 0$. In the basis in which θ_{ij} is given by equation (3.2), we have

$$n_{ij} = \text{diag} \left(0, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right). \quad (3.16)$$

The necessary condition for the correspondence principle to hold can then be stated more covariantly as

$$n_{ij} n_{kl} \hat{\theta}_{ik} = \theta_{jl} + \text{corrections that vanish for } N \rightarrow 0. \quad (3.17)$$

It follows that the vector $\epsilon^{i_1, i_2, \dots, i_d} \hat{\theta}_{i_1, i_2} \dots \hat{\theta}_{i_{d-2}, i_{d-1}}$ should be nearly parallel to the normal vector n_i . We conjecture that this vector is related to the total volume of the surface via

$$\text{Volume}_{d-1}(\text{emergent surface}) = C \text{Tr} \sqrt{\sum_{i_d} (\epsilon^{i_1, i_2, \dots, i_d} [X_{i_1}, X_{i_2}] \dots [X_{i_{d-2}}, X_{i_{d-1}}])^2}. \quad (3.18)$$

C in the above is some numerical coefficient which does not depend on N (for $d = 3$, this coefficient was 2π).

When interpreting the emergent surface as a higher-dimensional D-brane emerging from D0-branes via the dielectric effect [9], the two form $\hat{\theta}_{ij}$ and its pullback to the world-volume of the D-brane, θ_{ij} , will enter into the non-abelian BI and CS actions as expected. Finally, an emergent D-brane should have a U(1) connection living on its worldvolume; following [1], we can define it as

$$2v^i A_i = -iv^i \langle \alpha(x_i) | \partial_i | \alpha(x_i) \rangle, \quad (3.19)$$

where v^i is a tangent vector on the emergent surface. Working with a coherent state in a factorized form, we obtain that associated curvature is $\partial_{[i} A_{j]} = (\theta^{-1})_{ij}$, as expected.

4 Even dimensional spheres S^{2n} and noncommutative space with $SO(2n)$ invariance

The noncommutative four sphere can be constructed as in [10] (see also [11]). The starting point is a representation of the Clifford algebra in four dimensions: the γ matrices of section 1. The matrices in this representation act on vectors in a four-dimensional spinor representation. Consider then an irreducible representation of $\text{Spin}(5)$ given by the completely symmetric tensor product of k copies of this irrep. To each γ , associate a matrix X^i that acts on this tensor product as follows

$$X_i = \frac{1}{k} (\gamma^i \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \mathbf{1} \otimes \gamma^i \otimes \dots \otimes \mathbf{1} + \dots + \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes \gamma^i)_{\text{sym}} \quad (4.1)$$

The claim is that these five position matrices represent a four-sphere of radius one. Their dimension is $N = (k+1)(k+2)(k+3)/6$.

In the four dimensional spinor irrep, consider the the vector V_4 , and take its image under the k^{th} symmetric tensor product map, $(V_4 \otimes \dots \otimes V_4)_{\text{sym}} := (V_4)^{\otimes k}$. Because $\gamma^1 V_4 = V_4$ and $(\gamma^2 + i\gamma^3)V_4 = (\gamma^4 + i\gamma^5)V_4 = 0$, the matrices X_i above have a simple action on this vector,

$$X_1 \cdot (V_4)^{\otimes k} = (V_4)^{\otimes k}, \quad (4.2)$$

$$(X_2 + iX_3) \cdot (V_4)^{\otimes k} = 0, \quad (4.3)$$

$$(X_4 + iX_5) \cdot (V_4)^{\otimes k} = 0. \quad (4.4)$$

Consider the point $(1, 0, 0, 0, 0)$ in \mathbb{R}^5 , which we hope lies on the emergent sphere. We need the embedding operator E_5 at this point to have a zero eigenvector. We rewrite E_5 at this point as

$$\gamma^1 \otimes (X^1 - 1) + \sum_{a=1}^2 (\Lambda_-^a \otimes X_a^+ + \Lambda_+^a \otimes X_a^-), \quad (4.5)$$

where

$$X_a^\pm = X_{2a} \pm iX_{2a+1}. \quad (4.6)$$

Now, consider a vector $\Lambda = V_4 \otimes (V_4)^{\otimes k}$. It is easy to see that this is a zero eigenvector of E_5 as given in equation (4.5). Since the γ matrices form a fundamental (or standard) representation of $so(5)$, we recover the entire spherical surface with radius 1 by symmetry. However, using the methodology from section 3, rotational symmetry appears lost, as $\langle (V_4)^{\otimes k} | [X_2, X_3] | (V_4)^{\otimes k} \rangle = \langle (V_4)^{\otimes k} | [X_4, X_5] | (V_4)^{\otimes k} \rangle = 1/k$ and the other four commutators vanish. Since we know that the noncommutative sphere has $SO(5)$ symmetry and therefore $SO(4)$ symmetry once a point on the sphere is fixed, the noncommutativity on the 4-sphere must be of a different kind than that in section 3.

In fact, $\Lambda = V_4 \otimes (V_4)^{\otimes k}$ is not the only zero eigenvector of the embedding operator in equation (4.5). In contrast to the flat noncommutative space above, here both the raising and the lower operators X_a^\pm have zero eigenvectors. Let $\tilde{V}_4 = \Lambda_-^1 \Lambda_-^2 V_4$ be the spinor with $\gamma^1 \tilde{V}_4 = \tilde{V}_4$ and $\Lambda_-^a \tilde{V}_4 = 0$. Then, consider an arbitrary unit spinor W in the span

of $\{V_4, \tilde{V}_4\}$, $W = \mu V_4 - \nu \tilde{V}_4$, $|\mu|^2 + |\nu|^2 = 1$, with $\gamma^1 W = W$, $(\mu \Lambda_+^1 - \nu \Lambda_-^2)W = 0$ and $(\mu \Lambda_+^2 + \nu \Lambda_-^1)W = 0$. Rewrite the embedding operator in equation (4.5) as

$$\begin{aligned} & \gamma^1 \otimes (X^1 - 1) + \\ & (\mu \Lambda_+^1 - \nu \Lambda_-^2) \otimes (\mu X_1^- - \nu X_2^+) + (\nu \Lambda_+^1 + \mu \Lambda_-^2) \otimes (\nu X_1^- + \mu X_2^+) + \\ & (\mu \Lambda_+^2 - \nu \Lambda_-^1) \otimes (\mu X_2^- - \nu X_1^+) + (\nu \Lambda_+^2 + \mu \Lambda_-^1) \otimes (\nu X_2^- + \mu X_1^+) . \end{aligned} \quad (4.7)$$

This demonstrates explicitly that

$$\tilde{\Lambda} = W \otimes (W)^{\otimes k} \quad (4.8)$$

is also a zero eigenvector of the embedding operator in equation (4.5). The kernel of the embedding operator is a $(k+2)$ -dimensional space, while the space of the associated coherent states is $(k+1)$ -dimensional.³ Its presence has a natural interpretation: it is the auxiliary space necessary to ensure that the emergent noncommutative space has $SO(4)$ symmetry. (Notice that the noncommutative flat space we defined in the previous section *does not* have full rotational symmetry even when we set all θ_a equal to each other.)

To see how rotational invariance is restored, first notice that the $SO(4)$ symmetry we wish to see restored is generated by the six commutators $[X_i, X_j]$. For $k=1$, we write these commutators explicitly:

$$\begin{aligned} L_1 &:= -i[X_2, X_3] = \sigma_2(0, 3), \quad L_2 := -i[X_2, X_4] = \sigma_2(2, 1), \quad L_3 := -i[X_3, X_4] = \sigma_2(2, 2), \\ K_1 &:= -i[X_4, X_5] = \sigma_2(3, 0), \quad K_2 := -i[X_5, X_3] = \sigma_2(1, 2), \quad K_3 := -i[X_2, X_5] = -\sigma_2(1, 1). \end{aligned}$$

For larger k , we just consider these operators acting on the symmetric k^{th} tensor power of the four-dimensional irrep of $\text{Spin}(5)$. Notice that when one of these six generators acts on any vector in the kernel of the embedding operator, we get another vector in the kernel. Thus, we get a representation of the algebra $so(4)$. To see which representation it is, consider two mutually commuting sets of generators, $L_i \pm K_i$. Their commutation relationships are

$$[(L_i \pm K_i), (L_j \pm K_j)] = 2i\epsilon_{ijk}(L_k \pm K_k) \text{ and } [(L_i \pm K_i), (L_j \mp K_j)] = 0, \quad (4.9)$$

which is nothing more but the standard fact that $SO(4) \sim SU(2) \times SU(2)$. By explicit computation, we see that when acting on the kernel of the embedding operator, $L_i - K_i$ vanish, while the action of $L_i + K_i$ is that of a $(k+1)$ -dimensional irreducible representation of $su(2)$. Thus, the zero eigenvectors of the embedding operator form the $(k/2, 0)$ irrep of

³First, let's understand why the vectors $(W)^{\otimes m}$ span a $(m+1)$ -dimensional space (ie, why $\text{Sym}^m(\text{span}\{V_4, \tilde{V}_4\})$ is $(m+1)$ -dimensional), by drawing a parallel with representations of $SU(2)$. The fundamental irrep of $SU(2)$ is of course 2-dimensional, and all higher irreps correspond to completely symmetric tensor powers of the fundamental representation. Thus we know that the dimension of $\text{Sym}^m S$ where S is any two dimensional vector space is $m+1$. Thus, the kernel has dimension $k+2$ (because it corresponds to $W^{\otimes(k+1)}$), but there are only $k+1$ linearly independent N -dimensional coherent states once the first term in the product is stripped off.

$SU(2) \times SU(2)$. For example, for $k = 1$ we have, explicitly in the $\{V_4, \tilde{V}_4\}$ basis

$$\begin{aligned} -i[X_2, X_3] &= \sigma_2(0, 3) \rightarrow m_{23} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ -i[X_3, X_4] &= \sigma_2(2, 2) \rightarrow m_{34} := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \\ -i[X_2, X_4] &= \sigma_2(2, 1) \rightarrow m_{24} := \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}. \end{aligned} \quad (4.10)$$

So far, we have focused on the point $(1, 0, 0, 0, 0)$. However, when other points close enough to this one are considered, the commutators $[X_i, X_j]$ for $i, j = 2, \dots, 5$ are nearly constant. Consider, for example, a zero eigenvector of the embedding operator E_5 at a point $(1, \hat{\beta}, 0, 0, 0)$, with $\beta \ll 1$. Let's use a basis for the four dimensional spinor representation given by $\sigma_2(3, 0)|s_1, s_2\rangle = s_1|s_1, s_2\rangle$ and $\sigma_2(0, 3)|s_1, s_2\rangle = s_2|s_1, s_2\rangle$, where $s_i = \pm 1$. In this notation, $V_4 = |++\rangle$ and $\tilde{V}_4 = |--\rangle$. For clarity, pick an eigenvector of the embedding operator at point $(1, \hat{\beta}, 0, 0, 0)$ of the form

$$\begin{aligned} (|++\rangle + \beta|--\rangle + \dots)^{\otimes k} &= |++\rangle^{\otimes k} + k\beta(|+-\rangle \otimes |++\rangle^{\otimes(k-1)})_{\text{sym}} \\ &+ \frac{1}{2}k(k-1)\beta^2(|+-\rangle \otimes |+-\rangle \otimes |++\rangle^{\otimes(k-2)})_{\text{sym}} + \dots \end{aligned} \quad (4.11)$$

where β is proportional to $\hat{\beta}$. $(|+-\rangle \otimes |++\rangle^{\otimes(k-1)})_{\text{sym}}$ has length $1/\sqrt{k}$, $(|+-\rangle \otimes |+-\rangle \otimes |++\rangle^{\otimes(k-2)})_{\text{sym}}$ has length approximately $1/\sqrt{k(k-1)/2}$, etc. . . . Thus, $\sqrt{k}\beta$ is of order 1, this vector's overlap with $(|++\rangle)^{\otimes k}$ decreases sharply to zero.^{4,5} That a smooth sphere is recovered in the large k limit tell us that there is a range of values for β (or $\hat{\beta}$) where the vector above is close to being linearly independent of $(|++\rangle)^{\otimes k}$ but where terms with powers of β greater than some $p \ll k$ can be ignored. In this range, the matrix elements of $[X_i, X_j]$ when acting on the kernel of the embedding operator are approximately independent of β . As an example, consider that

$$\langle ++ |^{\otimes k} - i[X_3, X_4] (|+-\rangle \otimes |++\rangle^{\otimes(k-1)})_{\text{sym}} \quad (4.12)$$

is of order $1/k$, because the above overlap is only nonzero when the nontrivial operator in

$$-i[X_3, X_4] = \frac{1}{k} (\sigma_2(2, 2) \otimes \mathbf{1} \otimes \dots \otimes \mathbf{1} + \mathbf{1} \otimes \sigma_2(2, 2) \otimes \dots \otimes \mathbf{1} + \dots + \mathbf{1} \otimes \mathbf{1} \otimes \dots \otimes \sigma_2(2, 2))_{\text{sym}} \quad (4.13)$$

‘finds’ $|+-\rangle$ when acting on $(|+-\rangle \otimes |++\rangle^{\otimes(k-1)})_{\text{sym}}$.

Thus, for points near $(1, 0, 0, 0, 0)$, the relevant commutators, when acting on the kernel of the embedding operator, are nearly constant (with $1/k$ corrections) and we get the

⁴From our work [2], we would expect this overlap to have Gaussian fall-off.

⁵Thus, the radius of a noncommutative ‘cell’ is $1/\sqrt{k}$ and its 4-volume is $1/k^2$. In a sphere of radius 1, we then have approximately k^2 such ‘cells’. Each corresponds to a $k+2$ dimensional kernel of the embedding operator, so the total dimensionality of the matrices needs to be approximately k^3 , in agreement with the exact formula $N = (k+1)(k+2)(k+3)/6$.

following approximate noncommutative algebra

$$[X_i, X_j] = \frac{i}{k} m_{ij}, \quad (4.14)$$

where m_{ij} are $(k+1) \times (k+1)$ matrices in the $(k/2, 0)$ irreducible representation of $SU(2) \times SU(2)$. m_{23} , m_{24} and m_{34} are defined in equation (4.10), while $m_{25} = m_{34}$, $m_{35} = m_{24}$ and $m_{45} = m_{23}$. The factor $1/k$ comes from normalization of X_i in equation (4.1). This noncommutativity algebra is similar to spin noncommutativity with $SO(3)$ symmetry in three spacial dimensions in [12] (see also references therein).

$SO(4)$ is restored in equation (4.14) because the action of $SO(4)$ on X_2, X_3, X_4, X_5 can be ‘undone’ by a similarity transformation on matrices m_{ij} . Since $SU(2) \times SU(2)$ is a double cover of $SO(4)$, a rotation in \mathbb{R}^4 that goes ‘all the way around’ (ie, is trivial in $SO(4)$) corresponds to a nontrivial element of $SU(2) \times SU(2)$, namely $(-1) \otimes (-1)$. In the $(1/2, 0)$ irrep (and all $(k/2, 0)$ irreps for k odd) this corresponds to multiplying all the vectors in the kernel of the embedding operator by -1 . Such a change of basis has no effect on the matrix elements of $[X_i, X_j]$, or on m_{ij} . For $(k/2, 0)$ irreps with k even, $(-1) \otimes (-1)$ is trivial.

Another observation concerns orientability: a noncommutative 4-space with opposite orientation to the one we have considered is found at the other pole of the sphere, near the point $(-1, 0, 0, 0, 0)$. This can be obtained by taking $V_4 \rightarrow \tilde{V}_4$ and $\tilde{V}_4 \rightarrow V_4$. As such a map is not an element of $SU(2)$, it has a nontrivial effect on the matrices m_{ij} .

That the space of coherent states has dimension $k+1$ fits well with string theory: in [11] it was found that the correct interpretation of the four-sphere is that of a $D4$ -brane stack with k overlapping branes.⁶ Further, we notice that if we make a definition of a connection similar to that in equation (3.19), we will obtain a $U(k+1)$ gauge field, consistent with the interpretation of a stack of $k+1$ emergent D-branes. Finally, substituting our solution into equation (3.18) we get an answer of the form (numerical coefficient) $\cdot k + \mathcal{O}(1/k)$ corrections, again confirming that what we have obtained is a sphere of radius one, wrapped k (or $k+1$) times. This wrapping seems to be necessary to recover full rotational symmetry.

The string theory representation raises the following puzzle: is it possible to make a single emergent spherical $D4$ -brane? In [11] this puzzle was phrased differently: is it possible to separate the $k+1$ branes making up the stack and give them different radii? We take a partial step towards a positive answer in the next section by giving up local $SO(4)$ invariance.

The generalization to from the four sphere to higher even-dimensional spheres, S^{2k} is straightforward. These spheres are constructed in the same way as the four sphere, S^4 , by simply using the higher dimensional γ matrices (see, for example, [13] for a review). $SO(2k)$ symmetry around a point on S^{2k} will be restored in much the same way that $SO(4)$ symmetry was restored around a point on S^4 , leading to higher dimensional versions of the noncommutative algebra (4.14). Even-dimensional noncommutative spheres have a rich phenomenology (see for example [14]), which it would be interesting to explore from the point of view of our embedding operator.

⁶Up to corrections of order $1/k$, which explains the discrepancy between k and $k+1$.

5 More examples in $d = 5$

In this section, we consider two relatively simple co-dimension one hypersurfaces in \mathbb{R}^5 , one of which has the topology and the symmetries of $(S^2 \times S^2)/\mathbb{Z}_2$, and the other is a round S^4 whose $\text{SO}(5)$ symmetry is broken by noncommutativity.

To embed $(S^2 \times S^2)/\mathbb{Z}_2$ in \mathbb{R}^5 , we consider the equation

$$(1 - x_2^2 - x_3^2)(1 - x_4^2 - x_5^2) = x_1^2. \quad (5.1)$$

The noncommutative version of this hypersurface is given by

$$\begin{aligned} X_1 &= J_3^{(1)} \otimes J_3^{(2)}, \\ X_2 &= \mathbf{1} \otimes J_1^{(2)}, \\ X_3 &= \mathbf{1} \otimes J_2^{(2)}, \\ X_4 &= J_1^{(1)} \otimes \mathbf{1}, \\ X_5 &= J_2^{(1)} \otimes \mathbf{1}, \end{aligned} \quad (5.2)$$

where the matrices $J_i^{(a)} = L_i^{(a)}/j_a$, while $L_i^{(a)}$ form two irreducible representations of $\mathfrak{su}(2)$: $[L_i^{(a)}, L_j^{(a)}] = i\epsilon_{ijk}L_k^{(a)}$, each with spin j_a , $a = 1, 2$. It is easy to see that, in the large spin limit, these matrices satisfy equation (5.1).

At the point $(1, 0, 0, 0, 0)$, the corresponding embedding operator has two zero eigenvectors, $V_4 \otimes (|j_1\rangle \otimes |j_2\rangle) := V_4 \otimes |\alpha_1\rangle$ and $\tilde{V}_4 \otimes (|-j_1\rangle \otimes |-j_2\rangle) := V_4 \otimes |\alpha_2\rangle$, where $J_3^{(a)}|m\rangle_a = m|m\rangle_a$. The local noncommutativity at this point is

$$\langle \alpha_1 | -i[X_2, X_3] | \alpha_1 \rangle = -\langle \alpha_2 | -i[X_2, X_3] | \alpha_2 \rangle = \frac{1}{j_1}, \quad (5.3)$$

$$\langle \alpha_1 | -i[X_4, X_5] | \alpha_1 \rangle = -\langle \alpha_2 | -i[X_4, X_5] | \alpha_2 \rangle = \frac{1}{j_2} \quad (5.4)$$

with the expectation values of the other commutators vanishing, and with all cross-terms between $|\alpha_1\rangle$ and $|\alpha_2\rangle$ vanishing as well for $j_i > 1/2$.

The set of matrices (5.2) has the expected $\text{SO}(3) \times \text{SO}(3)$ symmetry: an action of the symmetry group on the lower indices of $J_i^{(a)}$ is equivalent to a conjugation. However $\text{SO}(3) \times \text{SO}(3)$ is not a subgroup of $\text{SO}(5)$, so different points on the emergent surface are not equivalent and we cannot use symmetry to study zero eigenvectors of the embedding operator. Instead, we must resort to numerical analysis. Preliminary numerical study at various small spins (at most 2) shows that the emergent surface gets closer to that in equation (5.1) for larger matrices, and that the embedding operator has two zero eigenvectors everywhere on the emergent surface. This would imply that the emergent surface locally looks like a direct sum of two noncommutative spaces described in section 3. It is possible that there are some points of enhanced symmetry, though we did not find any. That we get two copies of noncommutative flat space locally is consistent with $S^2 \times S^2$ being a double-cover of the surface in equation (5.1).

A different noncommutative surface is given by

$$\begin{aligned}
 X_1 &= J_3^{(1)} \otimes J_3^{(2)}, \\
 X_2 &= J_3^{(1)} \otimes J_1^{(2)}, \\
 X_3 &= J_3^{(1)} \otimes J_2^{(2)}, \\
 X_4 &= J_1^{(1)} \otimes \mathbf{1}, \\
 X_5 &= J_2^{(1)} \otimes \mathbf{1}.
 \end{aligned} \tag{5.5}$$

These five matrices satisfy, in the large spin limit, the equation $\sum_i X_i^2 = 1$. Again, at the point $(1, 0, 0, 0, 0)$, $V_4 \otimes (|j_1\rangle \otimes |j_2\rangle)$ and $\tilde{V}_4 \otimes (|-j_1\rangle \otimes |-j_2\rangle)$ are zero eigenvectors of the corresponding embedding operator. At this point, the noncommutativity is the same as in the previous example. Since $\text{SO}(5)$ symmetry here is broken to $\text{SO}(3) \times \text{SO}(2)$, to study the whole surface, we resort to numerical analysis, which shows that the embedding operator has two eigenvectors at nearly all points on the sphere $\sum_i x_i^2 = 1$, except on the circle $x_1 = x_2 = x_3$, where the degeneracy is $2j_2 + 2$. We can explain the enhanced degeneracy on the circle as follows: on this circle, let's take (without loss of generality) the point $(0, 0, 0, 1, 0)$. The operator $\Lambda_-^2 \otimes \mathbf{1} + \mathbf{1} \otimes L_+^{(2)}$ commutes with $E_5(0, 0, 0, 1, 0)$ and generates a basis for its kernel when acting on $(|\sigma_1, +1\rangle \otimes |\sigma_3, -1\rangle) \otimes (|L_1^{(1)}, +j_1\rangle \otimes |L_3^{(2)}, -j_2\rangle)$ where the notation $|L, l\rangle$ means an eigenvector of operator L with eigenvalue l . Our interpretation is that this corresponds to a stack of two noncommutative spherical surfaces which ‘merge’ on the circle $x_1 = x_2 = x_3$ where, perhaps, the full $\text{SO}(4)$ symmetry is locally restored. Away from this circle, noncommutativity breaks $\text{SO}(4)$ symmetry while the surface is still a round sphere independent of matrix size.

These two examples illustrate the rich noncommutative phenomenology that can be studied using our embedding operators.

6 Even dimensions

In this section, we try to use dimensional reduction of our embedding operator E_d to obtain an embedding operator in even dimensions. Since it is possible to obtain matrix configurations corresponding to odd-dimensional noncommutative spaces (such as odd-dimensional spheres), it makes sense to explore even d from a geometric perspective even if the string theoretic interpretation would be unclear. However, we find that this naive attempt does not produce an embedding operator compatible with the usual construction of the noncommutative three sphere S^3 . Therefore, we leave even dimensional spaces for future work.

To obtain a guess for the embedding operator in even dimensions, simply assume that $X_d = \mathbf{0}$ in equation (1.2):

$$E_d(x_1, \dots, x_d) = \sum_{i=1}^{d-2} (\sigma^3 \otimes \tilde{\gamma}^i) \otimes (X_i - x_i) - (\sigma^1 \otimes \mathbf{1}) \otimes (X_{d-1} - x_{d-1}) - (\sigma^2 \otimes \mathbf{1}) \otimes (x_d) \tag{6.1}$$

It is possible to show that this operator has an eigenvector with eigenvalue zero only if $x_d = 0$ and if another operator, which we would like to identify with E_{d-1} , has an eigenvector with eigenvalue zero. This would lead us to propose that

$$E_{d-1} = \sum_{i=1}^{d-2} \gamma^i \otimes (X_i - x_i) + i\mathbf{1} \otimes (X_{d-1} - x_{d-1}) , \quad (6.2)$$

where the γ matrices are those for dimension $d-1$, is a suitable embedding operator in an even dimension $d-1 = 2n$. The last term can, equivalently, have a minus sign in front of it. Notice that the above embedding operator is not hermitian: this is inconvenient but seems unavoidable. A potentially interesting observation is that if we take the last dimension, d , to be time, then the matrix X_d would be anti-hermitian and E_d itself would be hermitian.

The most natural place to test this embedding operator is to take $d-1 = 4$ and try the matrices corresponding to a noncommutative S^3 (see, for example, [13, 15]). The corresponding embedding operator does not seem to have any eigenvectors away from the origin $(0,0,0,0)$. In particular, for the two smallest presentations of S^3 , with $N = 4$ and $N = 12$, when the corresponding embedding operator is evaluated at a point (x, y, z, w) , its determinant is $r^6(r^2 + 8)$ for $N = 4$ and $r^{16}(r^2 + 6)(r^2 + 4)^3$ for $N = 12$. We have normalized our matrices so that their largest eigenvalue is 1, and $r^2 = x^2 + y^2 + z^2 + w^2$. Clearly, the embedding operator has zero eigenvectors only at the origin $r = 0$. Thus, (6.2) does not seem to be the correct operator.

To understand why the embedding operator in equation (6.2) does not have the right properties, it is useful to look at equation (3.12). Let the X_i be a series of representations of some Lie algebra (such as $su(2)$ for the two-sphere), so scaled that eigenvalues have a fixed range. Due to this scaling, the commutators on the right hand side of equation (3.12) get smaller as the matrices grow. This, in turn, guarantees that the width of the coherent state, on the left hand side of equation (3.12), approaches zero as the matrices get large. However, when the embedding operator in equation (6.2) is squared, the off-diagonal terms fail to arrange themselves into commutators, and we cannot make any conclusions about the size of the coherent state. In work [16], the existence of coherent states whose width approaches zero as the matrices grow large was used to define an emergent surface in any number of dimensions at infinite N (but not at finite N , in contrast to our work). We suspect that the current embedding operator must lead to an equation similar in structure to (3.12), and this is why (6.2) fails.

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